Math 261B Tues. Nov. 10

Ex. $PGL_n/\mathbb{Z}$

$= PSL_n/\mathbb{Z}$

$$\mathcal{O}_{\mathbb{Z}}(PGL_n) = \left(\mathcal{O}_{\mathbb{Z}}(GL_n)\right)_0$$

$$\cong$$

$$\mathcal{O}_{\mathbb{Z}}(PSL_n) = \mathcal{O}_{\mathbb{Z}}(SL_n)_{n\mathbb{Z}}$$

$$\underline{PGL_n}(\mathbb{F}_q) = \mathrm{Hom}_{\mathbb{Z}\text{-algs}}\left(\mathcal{O}_{\mathbb{Z}}(PGL_n), \mathbb{F}_q\right)$$

$$\parallel$$

$$GL_n(\mathbb{F}_q)/\mathbb{F}_q^\times \leftarrow$$

(by Lang's Theorem)

$$\neq$$

$$\text{“}PSL_n(\mathbb{F}_q)\text{”} = SL_n(\mathbb{F}_q)/\mu_n(\mathbb{F}_q)$$

$K = \bar{K}$

$PGL_n(K) = GL_n(K)/K^\times$

$\uparrow$

$\mathbb{G}_m(K) = GL_1(K)$

$$\begin{tikzcd}
1 \arrow[r] & SL_n(\mathbb{F}_q) \arrow[r] & GL_n(\mathbb{F}_q) \arrow[r, "\det"] \arrow[d, squiggly] & \mathbb{F}_q^\times \arrow[r] & 1 \\
1 \arrow[r] & SL_n(\mathbb{F}_q)/\mu_n(\mathbb{F}_q) \arrow[r] & PGL_n(\mathbb{F}_q) \arrow[r] & \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^n \arrow[r] & 1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F}_q^\times \cdot I_n \arrow[r, "\det"'] & \mathbb{F}_q^\times \\
c\, I_n \arrow[r, mapsto] & c^n
\end{tikzcd}$$

$\nwarrow \quad \nearrow_{\neq}$ if $(\mathbb{F}_q^\times)^n \neq \mathbb{F}_q^\times$

$SL_n \to PSL_n$

$Q \subset X \subset (Q^\vee)^*$

$Z(G) \cong (X/Q)^*$

How to get finite simple groups:

SL_n

$X = (Q^\vee)^* \qquad X = Q$

$X^* = Q^\vee$

take simply connected, "simple" alg. groups

G/K

$G(\mathbb{F}_q)/Z(\mathbb{F}_q)$

will be a finite simple group (except in char p for a few bad primes)

Ex. More $\qquad \mathbb{F}_q \subset K = \bar{K}$

$G = G(K)$ with $\mathcal{O}_K(G) \cong K \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(G)$

$(a+b)^p = a^p + b^p$

in char k = p

(Relative) Frobenius morphism:

$$G \xrightarrow{F_q} G$$

$$\mathcal{O}_K(G) \leftarrow \mathcal{O}_K(G)$$

$$f^q \mapsfrom f \quad \text{for } f \in \mathcal{O}_K(G)$$

In coordinates $g = (g_{ij}) \longmapsto (g_{ij}^q)$

$$\begin{tikzcd}
\underline{\underline{G}}(\mathbb{F}_q) \arrow[d, equal] & = & \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathcal{O}_{\mathbb{Z}}(G), \mathbb{F}_q) \arrow[d, phantom, "\cap"] \\
G(K)^{F_q} & & G(K)
\end{tikzcd}$$

$\mathbb{F}_q = K^{F_q}$

More generally, suppose $\Phi : G \to G$ such that $\Phi^m = F_q$

$G^{\Phi} \subseteq \underline{\underline{G}}(\mathbb{F}_q)$

Particular example $SU_n(\mathbb{F}_q)$

$SU_n = \{ g \in SL_n(\mathbb{C}) \mid (g^*)^{-1} = g \}$

$\mathbb{R}, \mathbb{C}$ $\qquad g^* = \overline{g^T}$

$\mathbb{R} \subset \mathbb{C} \longleftrightarrow \mathbb{F}_q \subset \mathbb{F}_{q^2}$

analogous to $SU_n \subset SL_n(\mathbb{C})$

group of matrices preserving a pos. def. Hermitian $(\,,\,)$

$(v, w) = v^T \overline{w}$

SU_n is a compact real Lie group, not an alg. group over $\mathbb{C}$!

$K^{F_q} = \mathbb{F}_q$

$F_q^2 = F_{q^2} \qquad K^{F_q^2} = \mathbb{F}_{q^2}$

$F_q \curvearrowright \mathbb{F}_{q^2}$ has order 2

$F_q^2 = 1$ on $\mathbb{F}_{q^2}$

$$\{ g \in Sl_n(K) \mid F_q(\underset{\uparrow}{g^T})^{-1} = g \} \quad \text{``}\overset{\shortparallel}{\div}\text{''} \quad \overset{\text{def}}{=} \quad SU_n(\mathbb{F}_q)$$

$$= Sl_n(K)^{\Phi}$$

$$\Phi(g) = F_q(g^T)^{-1} \qquad \Phi^2 = F_{q^2}$$

$v^T A \bar{w}$

$A = A^*$,

pos. definite

$\vdots$

$$g^R = J_n g^T J_n$$

Instead use $\Phi(g) = F_q\,(g^R)^{-1}$, get $Sl_n(K)^{\Phi}$ conjugate by J_n to previous case.

Now usual $\underset{\overset{\cup}{T}}{B} \subset Sl_n$ is Φ stable

$$\Phi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \begin{pmatrix} t_n^{-q} & & \\ & \ddots & \\ & & t_1^{-q} \end{pmatrix} \qquad \begin{matrix} T \to T \\ X(T) \leftarrow X(T) \end{matrix}$$

Φ acts on $X(T)$: $\quad (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \quad \longleftrightarrow \quad t_1^{\lambda_1} \cdots t_n^{\lambda_n}$

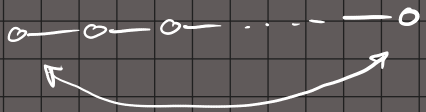
$$\begin{matrix} \downarrow \\ -q(\lambda_n, \dots, \lambda_1) \end{matrix} \qquad t_n^{-q\lambda_1} \cdots t_1^{-q\lambda_n}$$

$$\alpha_i = e_i - e_{i+1} = (\cdots 1 \ -1 \cdots)$$

$$(\cdots \ -1 \ \ 1 \cdots)$$

$$q \ \ -q$$

$$\begin{tikzcd} \alpha_i \arrow[d, mapsto] \\ q\,\alpha_{n-i} \end{tikzcd}$$



$\Phi^m = F_{q^m}$ Φ acts on simple roots by $\alpha_i \mapsto q\,\alpha_{\rho(i)}$ for some Dynkin diagram automorphism. ρ

Cartan Matrix + Diagram automorphism ρ + q $\longrightarrow$ Finite group $G(k)^{\Phi}$ of Lie type.

Quantum Groups

Quantum SL_2 , Quantum $B \subset SL_2$. Char $k = 0$ $(k = \mathbb{Q}(q))$ Later

$$B = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \in SL_2 \right\}$$

$\mathcal{O}(B) = k[t^{\pm 1}, x]$ $\mathcal{O}_{\mathbb{Z}}(B) = \mathbb{Z} \cdots$

$\Delta t = t \otimes t$

$$\left\{ \begin{aligned} \Delta x &= x \otimes t^{-1} + t \otimes x \\ S(t) &= t^{-1} \qquad S(x) = \ldots \end{aligned} \right.$$

$$\begin{pmatrix} t_1 & x_1 \\ 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} t_2 & x_2 \\ 0 & t_2^{-1} \end{pmatrix} = \begin{pmatrix} t_1 t_2 & t_2^{-1} x_1 + t_1 x_2 \\ 0 & (t_1 t_2)^{-1} \end{pmatrix}$$

$$\mathcal{U}(\mathfrak{b}) \qquad \mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = aH + bE \right\}$$

$$\left\{ \begin{aligned} &\overset{\shortparallel}{k\langle H, E \rangle} \big/ ([H,E] - 2E) \qquad [H,E] = 2E \\ &\Delta H = H \otimes 1 + 1 \otimes H \\ &\Delta E = E \otimes 1 + 1 \otimes E \\ &S(H) = -H, \qquad S(E) = -E \end{aligned} \right.$$

(Exercise : $[\ ,\ ]$ of primitives is primitive)

$$\mathfrak{b} \subset \mathcal{O}(B)^*$$

$$\mathcal{U}(\mathfrak{b}) \subset \mathcal{O}(B)^*$$

$$H \qquad \begin{aligned} 1 &\longmapsto 0 \\ t-1 &\longmapsto 1 \\ x &\longmapsto 0 \\ \mathfrak{m}_e \to (t-1, x)^2 &\longmapsto 0 \end{aligned} \qquad\qquad E \qquad \begin{aligned} 1 &\longmapsto 0 \\ t-1 &\longmapsto 0 \\ x &\longmapsto 1 \\ (t-1, x)^2 &\longmapsto 0 \end{aligned}$$

$$\frac{\partial}{\partial t}\Big|_I \qquad\qquad\qquad \partial/\partial x\Big|_I$$

$H: \quad f(t,x) \longmapsto \frac{\partial}{\partial \varepsilon} f(e^{\varepsilon}, 0)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

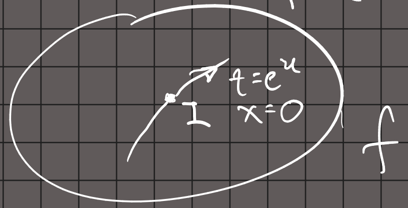
$= f_t(e^{\varepsilon}, 0) \cdot e^{\varepsilon}$

$\exp(\varepsilon H) = \begin{pmatrix} e^{\varepsilon} & \\ & e^{-\varepsilon} \end{pmatrix}$

$\nearrow$

$t\, f_t(t, 0) \cdot$

$t \frac{\partial}{\partial t} f$



$t^a x^m \qquad a \in \mathbb{Z} \qquad m \in \mathbb{N}$

$H: \quad t^a \longmapsto a$

$\text{others} \longmapsto 0$

$E: \quad t^a x \longmapsto 1$

$\text{others} \longmapsto 0$

$\mathcal{O}(B) = k[t^{\pm 1}, x]$

Δ

$H \cdot E \qquad t^a x^m \xrightarrow{\Delta} (t \otimes t)^a (x \otimes t^{-1} + t \otimes x)^m$

$= (t \otimes t)^a \sum_k \binom{m}{k} x^k t^l \otimes t^{-k} x^l$

$l = m - k$

$(k=0, l=1) \qquad H \otimes E \downarrow \qquad \downarrow E \otimes H \qquad (k=1, l=0)$

$m=1$

$t^a x \mapsto a+1 \qquad t^a x \longmapsto a-1$

$\text{others} \mapsto 0 \qquad \text{others} \mapsto 0$

$$\begin{array}{c}[H, E] \\ \shortparallel \\ 2E\end{array}$$

$t^a x \longmapsto (a+1) - (a-1) = 2$

others $\longmapsto 0$